28.Mathematical Delights from Indian Knowledge System

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Abstract

Indian knowledge system is a repository of the traditional knowledge created by Indians in a variety of domains like History, Basic Sciences, Culture, Art, Spirituality etc.
In the present article, we restrict our attention to the contributions in the field of Mathematics. Even a cursory look at the work carried out by Indian mathematicians immediately reveals that the contribution is huge and astonishing. We find a substantial advancement made by Indians in several branches of mathematics such as astronomy, algebra, geometry, arithmetic, indeterminate equations, calculus etc. Here, we focus on a few specific methods devised by Indians which display a great deal of originality. In particular, the article includes the Kuttaka method by Aryabhata, Chakraval method by Bhaskara-II, infinite series expansion for pi derived by Madhava. Finally, we shall highlight the legendary contributions by Ramanujan, the greatest mathematical genius that the world has ever seen!
In the process, we shall underline the methodology adopted by the Indian mathematicians and also exhibit the continuity in the works of mathematicians mentioned above and the aspect of chronology and historical credits for the inventions by way of several illustrations.
keywords: *Kuttaka method, Chakraval method, infinite series, Aryabhata-I, Bhaskaracharya-II, Madhava, Kerala School*

1. Introduction

The subject matter of this paper hinges around the contributions of ancient Indian mathematicians. In the labyrinth of Indian Knowledge System (IKS), the ingenuity, clarity and authority displayed by mathematicians of the past is simply mind boggling. The legacy of mathematical knowledge goes back to vedic period and culminates into the work of Ramanujan, a miracle in the history of mathematics. This passage of more than 3000 years is full of mathematical delights in a variety of branches such as algebra, arithmetic, astronomy, geometry, calculus, etc. The researches carried out by Indians are distinctively marked by the methods and techniques they invented. In the sequel, we shall illustrate these techniques by way of elaborating on some specific methods. In particular, we shall see that the methods are direct, algorithmic, constructive and techniques are recursive in nature. The recursive techniques allowed our ancestors to deal with the infinitesimal processes such as differentiation and integration.

On the background of a huge amount of work in diverse areas, we have restricted ourselves to a few specific methods devised by our ancestors. The choice of material is judicious in the sense that these methods can be included in the present day curriculum and would definitely inspire the future generations. In the process, the authors had to make many compromises by not choosing their favourite topics.

In section 2, we discuss the Kuttaka method for solving a linear diophantine equation in two variables. The method was invented by Aryabhata-I in 6th century and further suitably modified by Bhaskaracharya (Bhaskara-II) in 12th century. In this section, we also present Chakraval (cyclic) method developed by Bhaskara-II which describes the solutions of so called Pell's equation $Nx^2 + 1 = y^2$.

Section 3 highlights the work of Madhava (originator of the Kerala School) in 14th century. The work, in a sense, is continuation of the work carried out by Bhaskara-II. Here, we shall present the infinte series expansion for π obtained by Madhava. The technique used in the proof vouches for the recursive indigenous technique coupled with a geometric flavour.

A tribute to the Indian school cannot be complete without having gone through the glimpses of the work by legendary Indian mathematician Ramanujan- a grand culmination of the Indian mathematical dynasty.

2. Kuttaka Method

Aryabhata-I (476-550) is a prominent figure in the history of ancient Indian mathematics. In his famous work 'Aryabhatiya' he made several orginal contributions. Here, we present Aryabhata's method for solving a linear indeterminate equation in two variables of the form $ax - by = \pm c$, where a, b, c are positive integers. The constant term $\pm c$ is called *ksepa*. We assume that a and b have no common divisor. The method is widely known as 'Kuttaka' (refer to [5]). We begin with a few illustrations of Kuttaka method.

Example 2.1 *Consider equation* $\frac{55x+4}{12} = y$. *The method essentially relies on expressing the g.c.d. of* a *and* b *as* $ax + by$, *where* x *and* y *are integers. Though the Euclidean algorithm was not available to Aryabhata, the algorithm is embeded in the Kuttaka method. The quotients appeared in this process are called partial quotients. We have*

$$55 = 12(4) + 7$$
$$12 = 7(1) + 5$$
$$7 = 5(1) + 2$$
$$5 = 2(2) + 1$$

Here, the partial quotients are 4,1,1,2. Place these partial quotients one below the other in a column to form the *valli* (vertical line). In this example, the number of partial quotients is even that is *valli* is even. (The algorithm needs to be suitably modified if valli is odd.) Below the partial quotients, we write *ksepa* and zero at the end. Starting from the bottom, we complete the second *valli* as shown below. This gives one solution i. e. values of x (*adhara*) and y (*uttara*).

4	92	$20(4) + 12 = 92$
1	20	$12(1) + 8 = 20$
1	12	$8(1) + 4 = 12$
2	8	$4(2) + 0 = 8$
4	4	
0	0	

We get one positive solution $x = 20, y = 92$. If we divide these values 92 (*uttara*) and 20 (*adhara*) respectively by 55 $(i.e. a)$ and 12 $(i.e. b)$, we get $92 = 55(1) + 37, 20 = 15(1) + 5$. Since the quotients are same that is 1 in both divisions, then clearly the remainders are the smallest positive integers satisfying the given equation. Thus, $x = 8, y = 37$ is the smallest positive solution of the given equation $55x - 12y = -4$. Then it can be seen that there are infinitely many solutions given by $x = 8 + 12t, y = 37 + 55t, t$ an integer. The rationale behind the method is quite easy, but we include the same here for the sake of completeness. Starting with the last non-zero remainder we have

$$1 = 5 - 2(2)$$
$$4 = 4(5) - 2(8)$$
$$= 4(5) - 8[7 - 5(1)]$$
$$= 5(12) - 8(7)$$
$$= 12[12 - 7(1)] - 8(7)$$
$$= 12(12) - 7(20)$$
$$= 12(12) - 20[55 - 12(4)]$$
$$= 12(92) - 20(55)$$

Note that in the back substitution process we get coefficients 8,12,20,92 which appeared in the second *valli*. Observe that the coefficient of 7 is same as the coefficient of 55 which is 20, the value of x. The last expression implies that the coefficient of 12 that is 92 is the value of y. Thus $x = 20, y = 92$ is one solution of the given equation.

We now discuss an example where *valli* is odd.

Example 2.2 *Consider equation* $\frac{53x+4}{12} = y$. *As in Example 2.1,* $53 = 12(4) + 5$

$$12 = 5(2) + 2$$
$$5 = 2(2) + 1$$

As can be seen below, the *valli* is odd.

4	88
2	20
2	8
4	4
0	0

Observe that, unlike in Example 2.1, the values *uttara*$y = 88$ and *adhara*$x = 20$ is not a solution of the given equation $53x - 12y = -4$. Applying the process of back substitution we observe,

$$1 = 5 - 2(2)$$
$$4 = 5(4) - 2(8)$$
$$= 5(4) - 8[12 - 5(2)]$$
$$= 5(20) - 8(12)$$
$$= (20)[53 - 12(4)] - 8(12)$$
$$= (20)(53) - 12(88).$$

Multiplying both sides by -1, we get $-4 = (53)(-20) - 12(-88)$. In this case, we get negative solutions $x = -20, y = -88$. To obtain positive integral solutions we apply the following process:

$$-4 = 53(-20) - 12(-88)$$
$$= 53[-12(2) + 4] + 12[53(2) - 18]$$
$$= 53(4) - 12(18).$$

Thus we get smallest positive integral solution $x = 4, y = 18$ of the given equation $53x - 12y = -4$. Then all solutions can be described as before.

Remark 2.3 *In example 2.1, the smallest positive solution was obtained directly. However, this may not be always the case. The readers are encouraged to apply Kuttaka method to examples of the type* $\frac{11x+94}{3} = y$ *and find suitable modification to obtain the smallest positive solution.*

Remark 2.4 *If for the equation* $\frac{ax+c}{b} = y$, valli is odd and c is negative, then the values uttara and adhara in the second valli are positive integral solutions of the given equation. If valli is even and c is negative, then the values uttara and adhara in the second valli are negative integral solutions of the given equation. From these values we obtain positive integral solutions of the given equation as discussed in Example 2.2.

We now discuss Chakraval (Cyclic) method for solving Pell's equation. This gives us an opportunity to present the work of great Indian mathematician Bhaskara-II (1114-1185). He was born in Vijjalavida near Chalisgaon in Maharashtra. His famous book Sidhantashiromani was divided into four part: Lilavati (rule of arithmetic), Bijaganita (on algebra, root extraction), Grahaganitadhya (motion of planets) and Golodhyaya (on calculation of sphere). Bhaskara-II also suggested a modification in Kuttaka method (when valli is odd). As we shall see below, Kuttaka method is used as an intermediate step in Chakraval method.

2.1 Chakraval (Cyclic) Method

Here, we discuss the Chakraval method of solving the quadratic indeterminate equations of the type $Nx^2 \pm k = y^2$, where N, k are positive integers. The Indian mathematicians called it *varg prakrti*. A particular case of the equation $Nx^2 + 1 = y^2$, is presently known as the Pell's equation. Brahmagupta (598-668) was the first Indian mathematician who deal with such equations. Brahmagupta (in his book Brahma Sphuta Siddhanta) discovered that: (i) an integral solution of $Nx^2 + k = y^2$ can always be found if $k = \pm 1, \pm 2, \pm 4$ and (ii) from an integral solution of the auxilliary equation $Nx^2 + k = y^2$ where $k = \pm 1, \pm 2, \pm 4$ one can always find an integral solution of $Nx^2 + 1 = y^2$. Capitalizing on the work carried out by Brahmagupta, Bhaskara-II invented the chakraval (cyclic) method to solve Pell's equation (refer to [1]), a landmark in the history of ancient Indian mathematics. Here, we present an illustration of the cyclic method. First consider the following lemma:

Lemma 2.5 *If* $Na^2 + k = b^2$, *then*

$$N\left(\frac{am+b}{k}\right)^2 + \frac{m^2-N}{k} = \left(\frac{bm+Na}{k}\right)^2, \text{ for any integer } m.$$

The proof easily follows by expanding the brackets.

Example 2.6 *Consider the equation* $52x^2 + 1 = y^2$. *Observe that* $52(1)^2 - 3 = 7^2$.*Applying the Lemma 2.5, for* $a = 1, k = -3, b = 7$, *we get* $52\left(\frac{m+7}{3}\right)^2 + \frac{m^2-52}{(-3)} = \left(\frac{7m+52}{3}\right)^2$.

Suppose $\frac{m+7}{3} = t$ is an integer. Then $m = 3t - 7$. If coefficients of m and t are sufficiently large, Kuttaka method is used to find values of m. In this example we get $m = 2,5,8,\cdots$ For $m = 8, m^2 - 52$ is numerically smallest. Hence by taking $m = 8$, we get $52(5)^2 - 4 = 36^2$.

Now consider the auxilliary equation $52x^2 - 4 = y^2$. Now by applying the tulya-bhavana method of Brahmagupta, if (a, b) is a solution of the equation $Nx^2 + k = y^2$, then $(2ab, b^2 + Na^2)$ is a solution of the equation $Nx^2 + k^2 = y^2$. Thus we have

N	x	y	k
52	5	36	-4
	5	36	-4
	2(5)(36)=360	$(36)^2 + 52(5)^2 = 2596$	16

Now $52(360)^2 + 16 = (2596)^2$. Dividing the equation by 16, we get $52(90)^2 + 1 = (649)^2$. Thus $x = 90, y = 649$ is the solution of the equation $52x^2 + 1 = y^2$.

In general , by a repeated application of Lemma 2.5, we arrive at a suitable auxillary equation in finite number of steps. Then integral solution of this auxillary equation is obtained by using Tulya-bhavana method. For the detailed analysis of Chakraval method and its comparison with the modern method (using continued fractions) reader is referred to [8].

3. Calculus in Kerala School

In this section, we give highlights of the work carried out by mathematicians in Kerala School of mathematics. Madhava (1350-1425) is regarded as the originator of the Kerala School and the tradition continued for about 250 years. The disciple of Madhava and his disciples in turn worked in a variety of areas of mathematics. Here, we shall discuss the work carried out in the field of mathematical analysis. The use of techniques in Calculus (differentiation and integration) in their work is astonishing in the light of their period (1350-1620) much before Newton and Leibnitz. The methods adopted in Kerala

School exhibit a continuation of the typical Indian tradition. The recursive Indian method reached its pinnacle in kerala School. Some important inventions by the Kerala school are: Sum of infinite geometric series, Binomial series expansion, Infinite series expansion for trigonometric functions, Infinite series for $\tan^{-1}x$, Instantaneous velocity of a Planet, Surface Area of a sphere.

We now turn our attention towards the reasearch on π by Madhava. He found the value of π correct up to 11 decimal places (This is mentioned in "Kriyakramakari" by Shankara).

माधवाचार्यः पुनः अतोप्यासन्नतमां परिघिसङ्क्यामुक्तवान्–
विबुधनेत्रगजहिहुताशनत्रिगुणवेदभवारणंबाहवः।
नवनिखर्वमिते वृतिविस्तरे परिघिमानमिदं जगुदुर्बुधाः।।

विबुध	:	33
नेत्र	:	2
गज	:	8
अहि (सर्प)	:	8
हुताशन (अग्नि)	:	3
त्रिगुण	:	3
वेद	:	4
भ (नक्षत्र)	:	27
वारण (हत्ती)	:	8
बाहू	:	2
नव निखर्व	:	9×10^{11}

$$\pi = \frac{\text{circumference}}{\text{diameter}}$$
$$= \frac{2827433388233}{9 \times 10^{11}}$$
$$= 3.141592653592$$

We now discuss the attempt by Madhava to find the value of π through infinite series. (This is mentioned in "Yuktibhasha" by Jyeshthdev)

व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते।
त्रिशरादी विषमसङ्ख्याभक्तम् ऋणं स्वं पृथक् क्रमात् कुर्यात् ।।

व्यास वारिधिनिहते - $4 \times$ *diameter*, विषमसंख्याभक्तम् - divide by odd numbers,

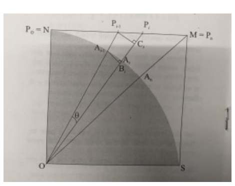
ऋणं स्वं पृथक् क्रमात - alternating in sign

$$\text{circumference} = 4 \times \text{diameter} \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

We now present the geometric construction given by Madhava (refer to [9]) to find this amazing infinte series which approximates π. The geometric construction demonstrates the mathematical ingenuity of Madhava.

If the diameter of a circle is D and the circumference is C, then by above series the value of π is $4 \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$.

Consider the square with side D and fit the circle with diameter D in the square as shown in the figure. (Only $1/4$ th part of the square is shown in the figure, OS=r=D/2.)



On the line segment NM, choose points $N = P_0, P_1, \cdots, P_n = M$ so that NM will be divided into n equal parts. If the length of line segment NM is r, then the length of each line segment $P_{i-1}P_i$ $(1 \le i \le n)$ is r/n. Since the circumference of the circle is C, the arc length of P_0A_n is $C/8$.

Madhava now used similarity of triangles.Observe that the triangles $OP_{i-1}C_i$ and $OA_{i-1}B_i$ are similar. So we have

$$\frac{A_{i-1}B_i}{OA_{i-1}} = \frac{P_{i-1}C_i}{OP_{i-1}}$$

Also by the similarity of triangles $P_{i-1}C_iP_i$ and OP_0P_i, we get

$$\frac{P_{i-1}C_i}{P_{i-1}P_i} = \frac{OP_0}{OP_i}$$

From the above two equations, we get

$$A_{i-1}B_i = \frac{(OA_{i-1})(OP_0)(P_{i-1}P_i)}{(OP_{i-1})(OP_i)} = (P_{i-1}P_i)\left(\frac{OA_{i-1}}{OP_{i-1}}\right)\left(\frac{OP_0}{OP_i}\right) = \left(\frac{r}{n}\right)\left(\frac{r}{k_{i-1}}\right)\left(\frac{r}{k_i}\right),$$

Here, we assume that the length of the line segment OP_i is k_i,

$1 \leq i \leq n$. Hence $A_{i-1}B_i = \left(\frac{r}{n}\right)\left(\frac{r^2}{k_{i-1}k_i}\right)$

The above equation is true for each value of $i = 1, \cdots, n$.

At this stage, Madhava mentioned that as the value of n increases the length of line segment $P_{i-1}P_i$ which is r/n decreases. Also the value of θ decreases and hence the value of $\sin\theta$ is approximately equal to θ. This is the key step of this geometric construction. Therefore we assume that the arc length of $A_{i-1}A_i$ is equal to the length of line segment $A_{i-1}B_i$. This is nothing but the linear approximation. Thus, the concept of differentiation later invented by Newton is very apparent in the work of Madhava. Now, we turn our attention towards finding the length of the circumference.

Recall that the arc length of P_0A_n is $C/8$. The arc P_0A_n consists of the arcs $A_0A_1, \cdots A_{n-1}A_n$. Therefore, $\frac{C}{8} = \sum_{i=1}^{n} A_{i-1}A_i$.

Since the arc length of $A_{i-1}A_i$ is equal to the length of line segment $A_{i-1}B_i$, we get,

$$\frac{C}{8} = \sum_{i=1}^{n} A_{i-1}B_i = \sum_{i=1}^{n} \left(\frac{r}{n}\right)\left(\frac{r^2}{k_{i-1}k_i}\right).$$

When value of n increases, the points P_{i-1} and P_i will be very close to each other and the difference between k_{i-1} and k_i will be very small. Hence we assume that $k_{i-1}k_i = k_i^2$. Thus we get,

$$\frac{C}{8} = \sum_{i=1}^{n} \left(\frac{r}{n}\right)\left(\frac{r^2}{k_i^2}\right) [I]$$

It is interesting to observe that the concept of integration is there in the above equation.

In general, observe that for real numbers x, y, z we may write

$$x\frac{y}{z} = x + x\frac{(y-z)}{z} = x + x\frac{(y-z)}{y}\frac{y}{z} = x + x\frac{(y-z)}{y} + x\frac{(y-z)}{y}\frac{(y-z)}{z}.$$

Continuing this process we get the following infinite series:

$$x\frac{y}{z} = x + x\frac{(y-z)}{y} + x\frac{(y-z)^2}{y^2} + x\frac{(y-z)^3}{y^3} + \cdots$$

If we put $x = r/n, y = r^2, z = k_i^2$ in the above series, then from equation $[I]$, we get

$$\frac{C}{8} = \sum_{i=1}^{n} \left[\frac{r}{n} - \frac{r}{n}\left(\frac{k_i^2 - r^2}{r^2}\right) + \frac{r}{n}\left(\frac{k_i^2 - r^2}{r^2}\right)^2 - \cdots\right]$$

But, $k_i^2 - r^2 = \left(i\frac{r}{n}\right)^2$.

$$\frac{C}{8} = \frac{r}{n}[1 + 1 + \cdots + 1] - \frac{r}{n}\frac{1}{r^2}\left[\left(\frac{r}{n}\right)^2 + \left(\frac{2r}{n}\right)^2 + \cdots + \left(\frac{nr}{n}\right)^2\right] + \frac{r}{n}\frac{1}{r^4}\left[\left(\frac{r}{n}\right)^4 + \left(\frac{2r}{n}\right)^4 + \cdots + \left(\frac{nr}{n}\right)^4\right]$$
$$- \frac{r}{n}\frac{1}{r^6}\left[\left(\frac{r}{n}\right)^6 + \left(\frac{2r}{n}\right)^6 + \cdots + \left(\frac{nr}{n}\right)^6\right] + \cdots [II]$$

If we take powers of r/n common from every bracket, then we get the sum of even powers of natural numbers in each bracket. The mathematicians from Kerala school knew the formulae for such sums $\sum_{i=1}^{n} i^k = \frac{n^{k+1}}{k+1}$. Substituting these values in equation $[II]$, we get

$$\frac{C}{8} = r\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right).$$

Thus,

$$C = 4D\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right).$$

Hence

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right).$$

The above series is called Newton-Gregory series.

The next section is devoted to the work of legendary Indian mathematician Shrinivasa Ramanujan. His work can be thought of as a culmination of the work carried out by Indian mathematicians.

4. Ramanujan

Shrinivasa Ramanujan (1887-1920) is surely the greatest mathematical genius that world has ever produced. Despite the lack of adequate facilities and substantial training in higher mathematics, he went on to produce miraculous mathematical masterpieces. He lived for only 32 years, but every moment he spent, was full of his passion for mathematics. Ramanujan was the most romantic figure in the recent history of mathematics, whose career seems full of paradoxes and contradictions.

Ramanujan mainly worked in analytic number theory and in the process exhibited unprecedented talent in the areas such as infinite series, continued fractions and nested radicals. Here, we have chosen a few illustrations of Ramanujan's work which can be appreciated even by lowbrows (refer to [2]).

4.1 Sum of cubes

Consider the Diophantine equation

$$X^3 + Y^3 = Z^3 + W^3$$

Observe that $1729 = 1^3 + 12^3 = 10^3 + 9^3$. Ramanujan gave a family of solutions as stated below:

Theorem 4.1 *If* $a^2 + ab + b^2 = 3\lambda c$, *then*

$$(a + \lambda^2 c)^3 + (\lambda b + c)^3 = (\lambda a + c)^3 + (b + \lambda^2 c)^3$$

For 1729, take $(a, b, c, \lambda) = (3,0,1,3)$.

After 1729, the next smallest such number is 4104. The problem of characterizing all positive integral solutions is still unsolved.

4.2 Sum of powers of 4

Consider the equation

$$a^4 + b^4 + c^4 + d^4 + e^4 = f^4$$

Example : $4^4 + 6^4 + 8^4 + 9^4 + 14^4 = 15^4$.

Ramanujan offered two families of solutions:

Theorem 4.2 *If* s, t, m, n *are arbitrary integers, then*

$$(8s^2 + 40st - 24t^2)^4 + (6s^2 - 44st - 18t^2)^4 + (14s^2 - 4st - 42t^2)^4 + (9s^2 + 27t^2)^4 + (4s^2 + 12t^2)^4 = (15s^2 + 45t^2)^4$$

and

$$(4m^2 - 12n^2)^4 + (3m^2 + 9n^2)^4 + (2m^2 - 12mn - 6n^2)^4 + (4m^2 + 12n^2)^4 + (2m^2 + 12mn - 6n^2)^4 = (5m^2 + 15n^2)^4.$$

4.3 Two remarkable Identities

1. If $ad = bc$, then $(a + b + c)^n + (b + c + d)^n + (a - d)^n = (c + d + a)^n + (d + a + b)^n + (b - c)^n$ where $n = 2$ or 4.
2. If $f_{2m}(x, y) = (1 + x + y)^{2m} + (x + y + xy)^{2m} - (y + xy + 1)^{2m} - (xy + 1 + x)^{2m} + (1 - xy)^{2m} - (x - y)^{2m}$, then

$$64 f_6(x, y) f_{10}(x, y) = 45 f_8^2(x, y)$$

The identity (1) above gives two equal sums of three biquadrates whereas (2) is not very easy and could be a special case of a general result not yet discovered.

4.4 Infinite Sequence of Nested Radicals

Ramanujan asked for the solutions of the system : $x^2 = y + a, y^2 = z + a$, and $z^2 = x + a$.
This naturally leads to

$x = \sqrt{a + y} = \sqrt{a + \sqrt{a + z}} = \sqrt{a + \sqrt{a + \sqrt{a + x}}} = \sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}}$ Taking signs into account, x is a solution of a polynomial of degree 8. We may choose signs and get an infinite sequence converging to x. For example if $a = 2$, then one gets the identity

$$2\sin\left(\frac{\pi}{18}\right) = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - \cdots}}}}$$

As a consequence of a similar problem posed by Ramanujan, one gets following identity which involves a nested sequence of radicals:

$$\sqrt{5 + \sqrt{5 + \sqrt{5 + \sqrt{5 - \sqrt{5 + \sqrt{5 + \cdots}}}}}} = \frac{1}{2}(2 + \sqrt{5} + \sqrt{15 - 6\sqrt{5}})$$

4.5 Approximations of π

Ramanujan was a master in the use of continued fractions. For example: $\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{293 + \cdots}}}}$

which gives successive approximations $\frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \cdots$

The following formula given by Ramanujan for the approximation of π is in some sense very unusual and difficult to guess :

$$\pi \approx \left(97\tfrac{1}{2} - \tfrac{1}{11}\right)^{1/4}$$

Ramanujan in his short span of career made tremendous contributions in Mathematics, in particular in Number Theory. His Notebooks still serve as a great source of inspiration and motivation for researchers across the globe. Several results are still open and many other are part of extremely deep theory.

5. Epilogue

The present article aimed at describing the glimpses of the work carried out by Aryabhata, Bhaskaracharya, Madhava and Ramanujan. The authors believe that there is a common thread in the Indian tradition and algorithmic approach in the methods adopted by them. We feel that Kuttaka and Chakraval methods can be included in the present curriculum at appropriate stages. Also the work on Calculus in the Kerala School deserves wider attention and therefore could be mentioned at the stage where Calculus is introduced for the first time.

The mathematics invented by ancient Indian mathematicians is enormous in size and extremely high in quality. Authors are aware of several very important inventions which are not included here. We hope that the present study would generate some interest in ancient Indian school of thought.

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